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COMMENT

Exact cluster size distributions and mean cluster sizes for the q-state bond-correlated percolation model

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Abstract. We show that the q-state bond-correlated percolation model (QBCPM), which is the percolation representation of the q-state Potts model (QPM), on the lattice without closed loops is equivalent to the bond random percolation model (BRPM) on the same lattice. Using such results and exact results for the BRPM on the linear and Bethe lattices, we obtain exact cluster size distributions and the mean cluster sizes S for the QBCPM on the linear and Bethe lattices. The mean cluster sizes obtained from this method are the same as those obtained by more tedious exact calculations. Near the critical point, the average number of m site clusters per site, n_m, for the QBCPM on the linear and Bethe lattices may be written in the scaling form for large values of m, which is the geometrical basis for the scaling laws of critical exponents.

Percolation (Broadbent and Hammersley 1957, Hammersley 1957, Essam 1973, 1980, Stauffer 1979, 1985, Stauffer *et al* 1982, Deutscher *et al* 1983) is a branch of statistical mechanics which has developed rapidly in recent decades. In the bond random percolation model (BRPM) on a lattice G of N sites and E nearest-neighbour (NN) bonds, each bond of G is occupied independently with probability p and sites connected by occupied bonds are defined to be in the same cluster. For p above the critical point p_c, a percolating (infinite) cluster appears. The average number of m site clusters per lattice site, n_m, and the mean sizes of finite clusters S for the BRPM on the linear lattice with $p<p_c=1$ are given by (Stauffer and Jayaprakash 1978, Reynolds *et al* 1977, Stauffer 1985)†

$$n_m=(1-p)p^{m-1}(1-p) \tag{1}$$

$$S=\frac{1+p}{1-p}. \tag{2}$$

For the BRPM on the Bethe lattice with coordination number z, the corresponding p_c, n_m and S in the interior of the lattice are given respectively by (Flory 1941, Fisher and Essam 1961, Nakanishi and Stanley 1980, Stauffer 1985)†

$$p_c=\frac{1}{z-1} \tag{3}$$

$$n_m=p^{m-1}(1-p)^{2+(z-2)m}\frac{2[m(z-1)]!}{m![m(z-2)+2]!} \tag{4}$$

$$S=\frac{1+p}{1-(z-1)p}. \tag{5}$$

† These authors might not consider BRPM, but it is easy to derive the following formulae for the BRPM from their results. We also call n_m, $1\leqslant m<\infty$, the cluster size distribution.

It has been shown by Kasteleyn and Fortuin (1969) and Fortuin and Kasteleyn (1972) that the q-state Potts model (QPM) (Potts 1952, Wu 1982) for $q \to 1$ corresponds to the BRPM and it has been shown by Hu (1983a, 1984a, b, c, d, 1986a, b, c) and proposed by Sweeny (1983) that the QPM for $q > 1$ corresponds to a q-state bond-correlated percolation model (QBCPM)†.

Based on the connection that the QBCPM on the lattices without closed loops is equivalent to the BRPM on the same lattices, which will be derived below, in this comment we will extend the results of equations (1)-(5) for the BRPM to the QBCPM. The mean cluster sizes obtained from this method are the same as those obtained by other more tedious exact calculations (Hu 1986b, Wang and Wu 1976). Near the critical point, i.e. $\varepsilon = |p - p_c| \ll 1$, the average number of m site clusters per site, n_m, for the QBCPM on the linear and the Bethe lattices may be written in the scaling form

$$n_m = n^{-\tau} f(m^{\sigma}\varepsilon) \tag{6}$$

for large values of m as in the case of the BRPM.

Following the notation and derivation of Hu (1984a, 1986a), we may write the partition function for the QPM on a lattice G of N sites and E nearest-neighbour (NN) bonds as follows:

$$\begin{aligned} Z_N &= \sum_{s_1 \cdots s_N} \exp\left(K \sum_{\mathrm{NN}} \delta(s_i, s_j)\right) \\ &= \sum_{G' \subseteq G} (\mathrm{e}^K - 1)^{b(G')} q^{n(G')} \\ &= \mathrm{e}^{KE} \sum_{G' \subseteq G} p^{b(G')} (1-p)^{E - b(G')} q^{n(G')} \end{aligned} \tag{7}$$

where $b(G')$ and $n(G')$ are the numbers of occupied bonds and clusters in G', respectively, and

$$p = 1 - \mathrm{e}^{-K}. \tag{8}$$

It follows from Euler's theorem that

$$n(G') = N - b(G') + l(G') \tag{9}$$

where $l(G')$ is the number of closed loops in G'. From (7) and (9), we find that Z_N may be written as

$$Z_N = q^{N-E} (\mathrm{e}^K + q - 1)^E \sum_{G' \subseteq G} \bar{p}^{b(G')} (1 - \bar{p})^{E - b(G')} q^{l(G')} \tag{10}$$

where

$$\bar{p} = \frac{\mathrm{e}^K - 1}{\mathrm{e}^K + q - 1}. \tag{11}$$

For the lattices without closed loops, e.g. the Bethe lattice (BG) and the one-dimensional linear lattices (LG), $l(G') = 0$ and the QBCPM on such lattices is equivalent to the BRPM with $\bar{p}$ of (11) as a bond probability. Such correspondence is true for the probability weight of each G'.

† See also Kaufman and Andelman (1984) and Larsson (1986). For other lattice models, see Hu (1983b, 1984e, 1985a, b, 1987a, b).

Using $\bar{p}$ of (11) to replace p of (1) and (2), we have

$$n_m = (1-\bar{p})\bar{p}^{m-1}(1-\bar{p}) \tag{12}$$

$$S = \frac{1+\bar{p}}{1-\bar{p}} = \frac{2e^K + q - 2}{q} \tag{13}$$

for the QBCPM on the linear lattice. Note that (12) is consistent with (3.12) and (3.15) of Hu (1986b) and (13) is consistent with (3.17), (3.18) and (3.34) of Hu (1986b). The equations of Hu (1986b) mentioned above are derived by procedures which are more tedious than the present method.

Using $\bar{p}$ of (11) to replace p of equations (3)-(5), we have

$$\bar{p}_c = \frac{1}{z-1} \tag{14}$$

$$n_m = \bar{p}^{m-1}(1-\bar{p})^{2+(z-2)m} \frac{2[m(z-1)]!}{m![m(z-2)+2]!} \tag{15}$$

$$S = \frac{1+\bar{p}}{1-(z-1)\bar{p}} \tag{16}$$

for the QBCPM on the Bethe lattice.

Wang and Wu (1976) considered an external magnetic field that couples with one component of the Potts spin in the q-state Potts model with the following partition function:

$$Z_{WW} = \sum_{s_1 \ldots s_N} \prod_{\langle ij \rangle} \exp\left(K\delta(s_i, s_j) + B \sum_i \delta(s_i, 1) \right). \tag{17}$$

Wang and Wu (1976) have found that in the interior of the Bethe lattice the magnetic susceptibility of the model defined by (17) is given by

$$\chi = \frac{q-1}{q^2} \frac{1+\bar{p}}{1-(z-1)\bar{p}} \tag{18}$$

for

$$\bar{p} < \frac{1}{z-1}. \tag{19}$$

Using subgraph expansion, Wu (1978) has shown that Z_{WW} may be written as

$$\begin{aligned} Z_{WW} &= \sum_{G' \subseteq G} (e^K - 1)^{b(G')} \prod_c [\exp(Bn_c) + q - 1] \\ &= e^{KE} \sum_{G' \subseteq G} p^{b(G')}(1-p)^{E-b(G')} \prod_c [\exp(Bn_c) + q - 1] \end{aligned} \tag{20}$$

where the product is over all clusters c in G' and p is given by (8). Equation (20) becomes equation (7) when $B=0$. Using (20) to calculate the magnetic susceptibility χ in the interior of the Bethe lattice, we find that

$$\chi = \frac{q-1}{q^2} S_B \tag{21}$$

for

$$\bar{p} < \frac{1}{z-1} \tag{22}$$

where S_B is the mean size of finite clusters for clusters defined in (7) and (20). Comparing (18) and (21), we have

$$S_B = \frac{1+\bar{p}}{1-(z-1)\bar{p}} \tag{23}$$

which is the same as that of (16) derived from the connection between the QBCPM and the effective BRPM on the Bethe lattice. It is clear that to derive (16) from (5) is simpler than to derive (18) and (21).

It has been postulated (Stauffer 1975) that the average number n_m of clusters per lattice site for the random percolation problem follows a scaling relation near p_c (i.e. $\varepsilon = |p - p_c| \ll 1$) and for large cluster sizes m

$$n_m \propto m^{-\tau} f(\varepsilon m^{\sigma}) \tag{24}$$

where σ and τ are two free exponents and f is a scaling function. The other critical exponents may be expressed in terms of σ and τ and scaling relations of critical exponents follows straightforwardly (Stauffer 1975, 1985). It is easy to show that for $\varepsilon \ll 1$ and $m \gg 1$, n_m of (1) may be written as

$$n_m = m^{-2} f(\varepsilon m) \tag{25}$$

with $f(x) = x^2 \mathrm{e}^{-x}$ and n_m of (4) may be written as (Nakanishi and Stanley 1980)

$$n_m \sim 2[2\pi(z-2)(z-1)^5]^{-1/2} m^{-5/2} \exp\{-(\varepsilon m^{1/2})^2 (z-1)[2(z-2)]^{-1}\}. \tag{26}$$

Therefore the scaling assumption (24) is valid for the random percolation problem on the linear and Bethe lattices. Hu (1986b, c) and Sweeny (1983) have postulated that the average number of clusters per lattice site for clusters defined in (7) also follows the scaling relation of (24). It follows from (11), (12), (15), (25) and (26) that such a scaling assumption is also valid for the QBCPM on the linear and Bethe lattices. Therefore on such lattices the scaling relations of critical exponents may be understood from the scaling laws of cluster size distributions, i.e. n_m.

In a recent paper Larsson (1986) used equation (7) to formulate a real space renormalisation group calculation method for the q-state Potts model (QPM). However, Larsson's method contains some inconsistency which may be removed by introducing a background energy factor on the left-hand sides of (3.2a) and (3.2b) in his paper (Hu 1986d, Hu and Chen 1987). Instead of using (7), we may also use equation (10) of the present comment to formulate an alternative real space renormalisation group calculation method for the QPM. The details of such methods and calculation results will be presented in a further paper (Hu and Chen 1987).

In summary, we show that the q-state bond-correlated percolation model (QBCPM) on the lattice without closed loops is equivalent to the bond random percolation model (BRPM) on the same lattice. Based on such a connection and the existing exact results for the BRPM, we easily obtain exact cluster size distributions and mean cluster sizes for the QBCPM on the linear and Bethe lattices, which are the same as those obtained by other more tedious methods and which support the scaling assumption for cluster size distributions.

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